# Mechanical Ideas in Geometry

**Tadashi F. Tokieda**

Traditionally, pure mathematics is expected to turn up unexpected uses in applied sciences of a later period. This article samples some uses of an applied science (mechanics) in pure mathematics of an earlier period (euclidean geometry).

We begin by presenting an instance of 'mechanical thinking' that is difficult to replace by purely mathematical reasoning.

**Problem.** *Can a convex polyhedron contain a point whose perpendicular projection on every face falls outside the face?*

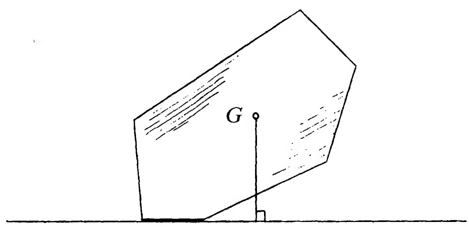


**Figure 1.** Perpendicular from $G$ falls outside *this* face.

The answer is no, because perpetual motion does not exist: by affixing a mass at such a point and leaving the rest of the polyhedron massless, we would have a tumbler that keeps working forever.

**I.** Probably the only theorem famous enough to elicit a nodding acknowledgment from all educated citizens is that of Pythagoras. Can we prove it by mechanics?

Let $PQR$ be a right triangle, with the right angle at $R$. Build a closed box with $PQR$ and its parallel copy as lid and bottom, and with side walls of height $h$.

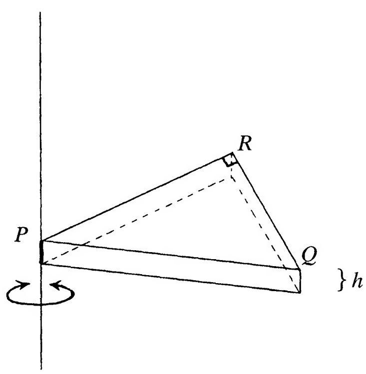


**Figure 2**

Hinge the box at corner $P$ to a vertical axis, around which the box can pivot freely (Figure 2).

Now, we fill this box with gas of pressure $p$.

The gas pushes against all five faces of the box. As the upward force on the lid cancels the downward force on the bottom, we consider only the forces on the sides. Here is a view from the top.

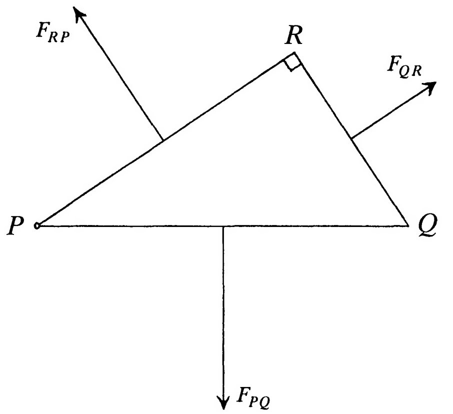


**Figure 3**

The three forces $F_{PQ}$, $F_{QR}$, $F_{RP}$ may be regarded as acting on the centers of the three sides. In Figure 3, $F_{PQ}$ tries to revolve the box clockwise, $F_{QR}$ and $F_{RP}$ counter-clockwise. To forestall perpetual motion,

$$\text{torque of } F_{PQ} \text{ about } P = |F_{PQ}| \cdot \frac{PQ}{2}$$

must balance

$$\text{torque of } F_{QR} = ? \qquad \text{and} \qquad \text{torque of } F_{RP} = |F_{RP}| \cdot \frac{RP}{2}.$$

But the stipulation that the angle $R$ be $90^\circ$ makes the torque of $F_{QR}$ come out $|F_{QR}| \cdot QR/2$. The balance condition therefore reads

$$|F_{PQ}| \cdot \frac{PQ}{2} = |F_{QR}| \cdot \frac{QR}{2} + |F_{RP}| \cdot \frac{RP}{2}.$$

Force is pressure times area, so $|F_{PQ}| = phPQ$, and similarly for $|F_{QR}|$ and $|F_{RP}|$. Dividing through by $ph/2$ we stumble upon

$$PQ^2 = QR^2 + RP^2.$$

**II.** More generally, if we repeat the same construction as in (I) but start from a triangle with an arbitrary angle at $R$, we arrive at the 'cosine law'. What about the 'sine law'?

On the frontispiece of his treatise *Practice of Weighing*, Stevin put the picture of Figure 4 with the caption 'Wonder, and is no wonder' [4]. The point is that the chain, although it carries unequal weights on the two slopes, does not keep slithering around the triangle.

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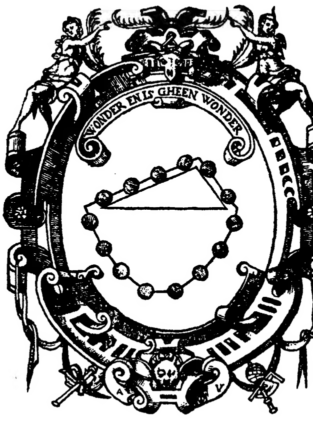


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**Figure 4**

Label the angles $\alpha$, $\beta$ and the slopes $a$, $b$ as shown (Figure 5). The amounts of mass on the slopes $a$, $b$ are proportional to the lengths of $a$, $b$, while the components of gravitational acceleration on the slopes $a$, $b$ are $g \sin \beta$, $g \sin \alpha$. Since the chain is immobile, the forces down the slopes $a$, $b$ balance:

$$ag \sin \beta = bg \sin \alpha, \qquad \text{or} \qquad \frac{\sin \alpha}{a} = \frac{\sin \beta}{b}.$$

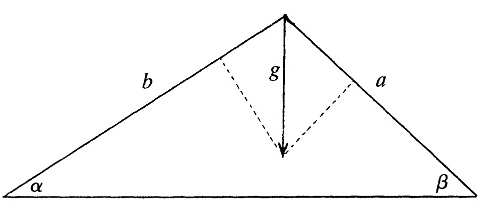


**Figure 5**

**III.** Next in fame in elementary mathematics are various centers of triangles. This section gives a mechanical treatment of some of them. The gas-filling trick will serve us well. In addition, we need a

**Lemma.** *If three planar forces keep a body in equilibrium, then* (i) *their vector sum is zero, and* (ii) *their lines of action meet in one point.*

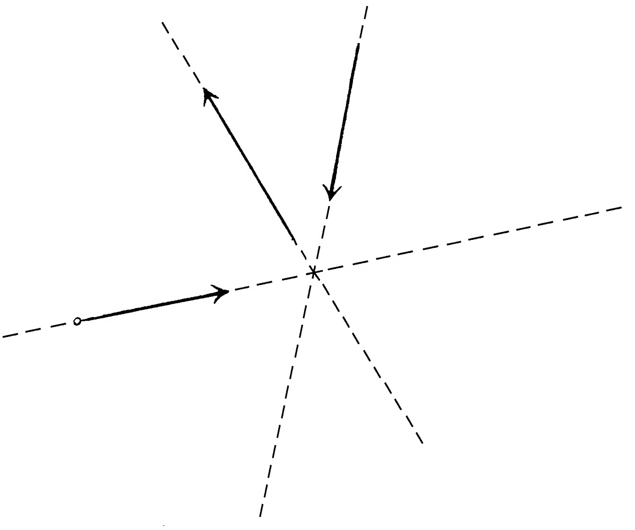

**Figure 6**

The assertions are easy to check: (i) means that the body does not translate, (ii) that it does not rotate (Figure 6).

(III-1) *The perpendicular bisectors of the sides of a triangle meet in one point* (*circumcenter*).

Manufacture a box from the triangle as before, and fill it with gas (no hinge this time). The forces from the gas act perpendicularly on the centers of the three sides; they are in equilibrium, lest the box should become a perpetual motion machine. By the lemma, their lines of action, which coincide with the perpendicular bisectors of the sides, meet in one point.

(III-2) *The perpendiculars dropped from the vertices of a triangle to their opposite sides meet in one point* (*orthocenter*).

This is in fact a paraphrase of (III-1): given a triangle $\Delta$, draw a dual triangle $\nabla$ whose sides contain the vertices of $\Delta$ and are parallel to the sides of $\Delta$. Then the putative orthocenter of $\Delta$ is none other than the circumcenter of $\nabla$.

(III-3) *The angle bisectors of a triangle meet in one point* (*incenter*).

Exert six forces of equal magnitude as in Figure 7. They are visibly in equilibrium. Combining the pair of forces at each vertex into a single force, we obtain three forces in equilibrium pointing along the angle bisectors. The lemma applies.

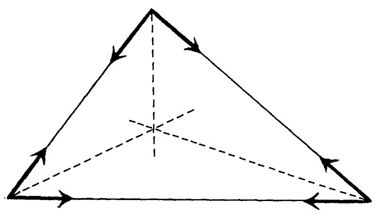

**Figure 7**

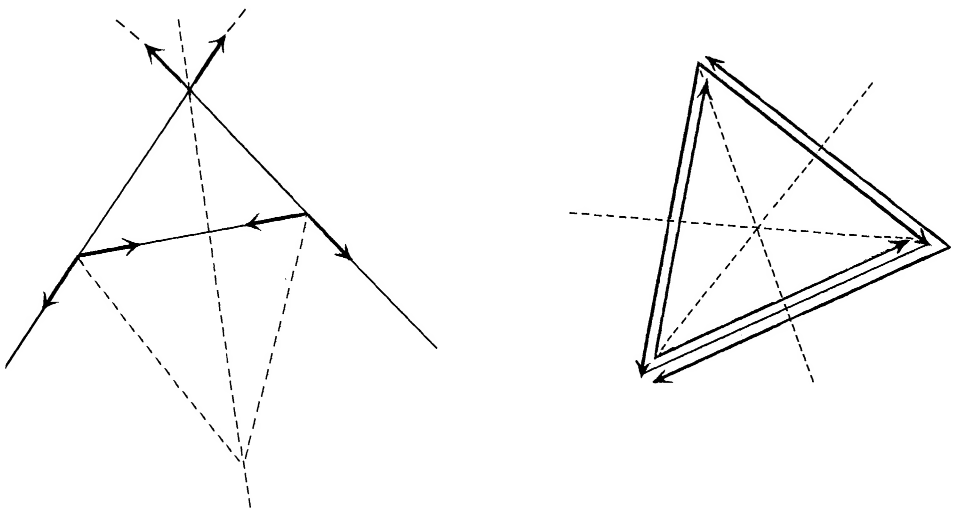

**Figure 8**

(III-4) *The internal bisector of one angle and the external bisectors of the other two angles of a triangle meet in one point* (*excenter*).
(III-5) *The medians of a triangle meet in one point* (*center of gravity*).

Again, exert the forces as in Figure 8.

**IV.** Unbeknownst to many, a triangle has yet another center, which may be called its 'short-center' (or 'brachycenter,' should a Hellenic pedigree be desired). In a triangle $ABC$, it is the point $S$ such that the sum of the distances $SA + SB + SC$ is minimal.

**Theorem.** *All three angles around the short-center are* $120°$.

This may look like a routine exercise in differentiation, but the calculation calls for some shrewdness. It can be established *à la* Euclid [**1**, pp. 21–22], but at the price of improbable ingenuity. All in all, the following proof by mechanics seems rather natural [**3**].

Set up a table with holes drilled in the positions of $A$, $B$, $C$, and three equal weights hanging on strings passing through the holes and whose ends are tied together in a knot (Figure 9).

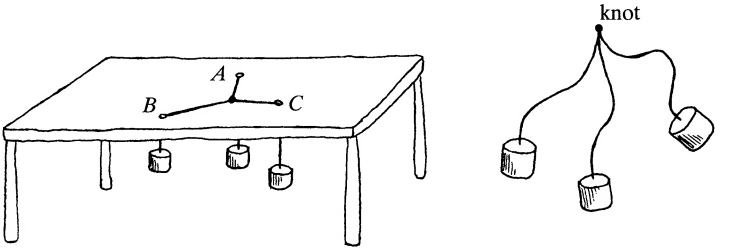


**Figure 9**

The weights maximize the total length of the strings under the table, or, what is the same thing, minimize it on the table. So the knot settles at $S$. Since the weights are equal, the strings are pulling at $S$ with equal strength. Now there is a plain but handy

**Principle.** *Three forces of equal magnitude acting on a common point are in equilibrium if and only if their heads form an equilateral triangle.*

It follows that the angles around $S$ are indeed all 120°.

You can witness this theorem in action in everyday life: generically, three soap films meet at 120°. (Locally the situation can be modeled by three coaxial planes minimizing the total area.)

**V.** The principle stated in (IV) makes perfect sense even without mechanical interpretation. We finish by applying it to an instance where 'mechanical thinking' is almost dispensable.

**Theorem** (a slight generalization of [**2**]). *Let $A_1A_2A_3$ and $B_1B_2B_3$ be equilateral triangles. If $X_i$ is the point that divides $A_iB_i$ in a fixed ratio $m:n$, then $X_1X_2X_3$ is also equilateral* (Figure 10).

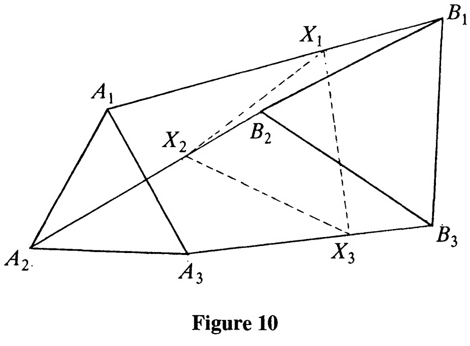


**Figure 10**

Denote by $O_A$ and $O_B$ the centers of $A_1A_2A_3$ and $B_1B_2B_3$, and by $O$ the point that divides $O_AO_B$ in the ratio $m:n$. Then

$$(\dagger) \qquad OX_i = \frac{n}{m+n}O_AA_i + \frac{m}{m+n}O_BB_i \qquad (i = 1,2,3).$$

On the one hand, squaring (†) we see that

$$|OX_i|^2 = \left(\frac{n}{m+n}\right)^2 |O_AA_i|^2 + \frac{2mn}{(m+n)^2}O_AA_i \cdot O_BB_i + \left(\frac{m}{m+n}\right)^2 |O_BB_i|^2$$

has the same value for all $i$. On the other hand, summing (†) over $i$ we get $OX_1 + OX_2 + OX_3 = 0$. Thus, if $OX_1$, $OX_2$, $OX_3$ are interpreted as three forces acting on $O$, they have equal magnitude and are in equilibrium. By the principle, their heads $X_1X_2X_3$ form an equilateral triangle.

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4. S. Stevin (van Brugghe), *De Weeghdaet*, Christoffel Plantijn Leyden, 1586.

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